Having found a value of mh from these equations, then α and ml are determined.

On examination, however, the second of these period equations will be found to have no real roots, while the only real roots of the first equation will be found to be given by $mh = 0$, repeated six times, giving the previous algebraical solution; this is seen by investigating the intersections of the curves $y = \cot x\sqrt{3}$ and $z = -\cosh x + 2\,\mathrm{sech}\, x$, or $y = 3\cos x\sqrt{3}$ and $z = -\cosh x - 2\,\mathrm{sech}\, x$, where $x = mh\sqrt{3}$.

If, however, we put

$$w = \sinh m(u-\alpha) - \sinh m(\beta u + \alpha) + \sinh m(\beta^2 u - \alpha),$$

then

$$\Phi = \sinh m(z-\alpha)\cos my - \sinh\tfrac{1}{2} m(y\sqrt{3} + z + 2\alpha)\cos\tfrac{1}{2} m(y - z\sqrt{3})$$
$$+ \sinh\tfrac{1}{2} m(y\sqrt{3} - z - 2\alpha)\cos\tfrac{1}{2} m(y + z\sqrt{3});$$

$$\Psi = -\cosh m(z-\alpha)\sin my + \cosh\tfrac{1}{2} m(y\sqrt{3} + z + 2\alpha)\sin\tfrac{1}{2} m(y - z\sqrt{3})$$
$$+ \cosh\tfrac{1}{2} m(y\sqrt{3} - z - 2\alpha)\sin\tfrac{1}{2} m(y + z\sqrt{3});$$

so that $\Psi = 0$, when $y = \pm z\sqrt{3}$, and the boundary conditions are satisfied.

Then, exactly as before, from the free surface condition

$$l\frac{\partial\Phi}{\partial z} = \Phi,$$

when $z = h$, for all values of y, we shall find

$$ml = \tanh m(h-\alpha), \tag{I}$$

$$\sin mh\sqrt{3} = \sqrt{3}\sinh m(h + 2\alpha), \tag{IV}$$

$$ml - \frac{1}{ml} = -\sqrt{3}\cot mh\sqrt{3}, \tag{V}$$

$$\sinh 2m(h-\alpha) = \frac{2}{\sqrt{3}}\tan mh\sqrt{3}, \tag{VI}$$

$$\sqrt{3}\sinh 3mh\cosh 2m\alpha = \sinh mh\sqrt{3}\left(\cosh 2mh + \frac{2\cosh mh}{\cos mh\sqrt{3}}\right),$$

$$\sqrt{3}\sinh 3mh\sinh 2m\alpha = \sinh mh\sqrt{3}\left(\sinh 2mh - \frac{2\sinh mh}{\cos mh\sqrt{3}}\right),$$

whence, eliminating α by squaring and subtracting, we obtain the period equation

$$\frac{3\sinh^2 3mh}{\sinh^2 mh\sqrt{3}} = \left(\cosh 2mh + \frac{2\cosh mh}{\cos mh\sqrt{3}}\right)^2 - \left(\sinh 2mh - \frac{2\sinh mh}{\cos mh\sqrt{3}}\right)^2$$
$$= 1 + \frac{4\cosh 3mh}{\cos mh\sqrt{3}} + \frac{4}{\cos^2 mh\sqrt{3}};$$

equivalent to

$$\cosh 3mh = -\cos mh\sqrt{3} + 2\sec mh\sqrt{3},$$

or

$$3\cosh 3mh = -\cos mh\sqrt{3} - 2\sec mh\sqrt{3}.$$

the new period equations, which by inspection have an infinite number of real roots.

Transfer the axis of x to the edge of the water on one bank, and expand w in descending powers of e^{mh}, retaining only the leading terms, supposing h is made infinite; we shall thus obtain Kirchoff's expression for the motion of standing waves on a shore sloping at 30° to the horizon (Gesammelte Abhandlungen, II, p. 434).

To do this we must begin by writing $y - h\sqrt{3}$ for y, and $z + h$ for z; and then

$$\begin{aligned}\Phi &= \sinh m(z + h - \alpha)\cos m(y - h\sqrt{3})\\ &- \sinh \tfrac{1}{2}m(y\sqrt{3} + z - 2h + 2\alpha)\cos\tfrac{1}{2}m(y - z\sqrt{3} - 2h\sqrt{3})\\ &+ \sinh\tfrac{1}{2}m(y\sqrt{3} - z - 4h - 2\alpha)\cos\tfrac{1}{2}m(y + z\sqrt{3}).\end{aligned}$$

Now, when h is indefinitely great, the period equation gives $\sec mh\sqrt{3} = \infty$, $\cos mh\sqrt{3} = 0$; so that

$$\begin{aligned}\Phi &= \sinh m(z + h - \alpha)\sin my\\ &- \sinh\tfrac{1}{2}m(y\sqrt{3} + z - 2h + 2\alpha)\sin\tfrac{1}{2}m(y - z\sqrt{3})\\ &+ \sinh\tfrac{1}{2}m(y\sqrt{3} - z - 4h - 2\alpha)\cos\tfrac{1}{2}m(y + z\sqrt{3}).\end{aligned}$$

Also, when $h = \infty$, $ml = 1$, and therefore $\tanh m(h - \alpha) = 1$, so that

$$\cosh m(h-\alpha) = \sinh m(h - \alpha) = \tfrac{1}{2}\exp m(h - \alpha) = C, \text{ suppose};$$

and, therefore, retaining the leading terms,

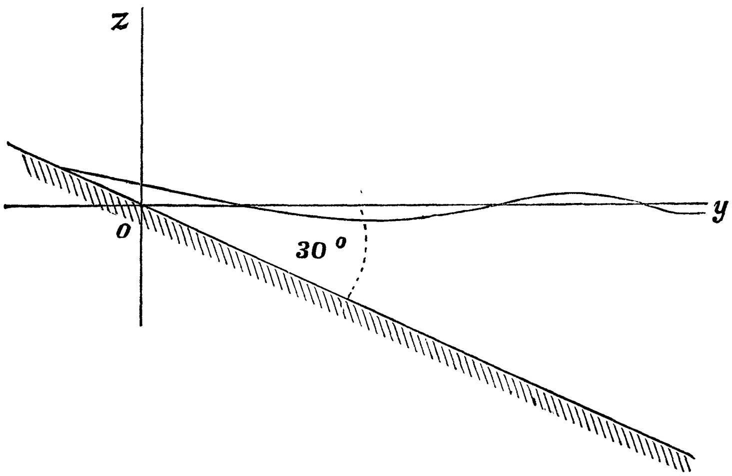
$$\begin{aligned}\Phi &= Be^{mz}\sin my\\ &- Be^{-\frac{1}{2}m(y\sqrt{3}+z)}\sin\tfrac{1}{2}m(y - z\sqrt{3})\\ &+ 2Be^{-m(h+2\alpha)}e^{-\frac{1}{2}m(y\sqrt{3}-z)}\cos\tfrac{1}{2}m(y + z\sqrt{3}).\end{aligned}$$

But from equation (IV)

$$\sin mh\sqrt{3} = 1 = \sqrt{3}\sinh m(h + 2\alpha) = \sqrt{3}\exp m(h + 2\alpha),$$

so that we may replace

$$2e^{-m(h+2\alpha)} \text{ by } \sqrt{3},$$



and then the value of Φ agrees with that given by Kirchoff, changing the sign of z, as our axis of z is drawn vertically upwards.

Then
$$\Psi = -Be^{mz}\cos my + Be^{-\frac{1}{2}m(y\sqrt{3}+z)}\cos\frac{1}{2}m(y - z\sqrt{3})$$
$$+ B\sqrt{3}e^{-\frac{1}{2}m(y\sqrt{3}-z)}\sin\frac{1}{2}m(y + z\sqrt{3});$$

so that
$$\Phi + i\Psi = B\exp\left(mz - \frac{1}{2}i\pi\right) + B\sqrt{3}\exp(m\beta u - i\pi) + B\exp\left(m\beta^2 u - \frac{3}{2}i\pi\right).$$

To verify that in this case, where $z = 0$,
$$\frac{\partial\Phi}{\partial z} = m\Phi,$$
it is convenient to notice that $\dfrac{\partial\Phi}{\partial z} = \dfrac{\partial\Phi}{\partial y}$; so that
$$\frac{\partial\Psi}{\partial y} = m\Phi,$$
where $z = 0$; and we may put $z = 0$ before differentiating with respect to y, which simplifies the work.

20. *Progressive Waves in a Channel of* 120°.

Just as from Kelland's solution for progressive waves in a channel of 90° we obtained the solution for standing waves across the channel by replacing the hyperbolic functions partially by circular functions, so, conversely, from the above solution for standing waves across a channel of 120°, we shall obtain the

solution for progressive waves by replacing the circular functions by the corresponding hyperbolic functions. Then, for progressive waves, we can put

$$\phi = A\Phi \cos\sqrt{2}\,(mx - nt),$$

where

$$\Phi = \sinh m(z-\alpha)\cosh my - \sinh\tfrac{1}{2}m(y\sqrt{3}+z+2\alpha)\cosh\tfrac{1}{2}m(y-z\sqrt{3})$$
$$+\sinh\tfrac{1}{2}m(y\sqrt{3}-z-2\alpha)\cosh\tfrac{1}{2}m(y+z\sqrt{3}),$$

and then
$$\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=2\Phi;$$

so that
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0;$$

so that the equation of continuity is satisfied.

We shall also find that when

(I)
$$y = z\sqrt{3},\ \frac{\partial\Phi}{\partial y}=\sqrt{3}\,\frac{\partial\Phi}{\partial z},$$

when

(II)
$$y=-z\sqrt{3},\ \frac{\partial\Phi}{\partial y}=-\sqrt{3}\,\frac{\partial\Phi}{\partial z};$$

so that the boundary conditions are satisfied.

To satisfy the free surface conditions, it is convenient to express Φ in the equivalent form

$$\Phi=\sinh m(z-\alpha)\cosh my-\sinh\tfrac{1}{2}m\{(\sqrt{3}+1)y+2\alpha\}\cosh\tfrac{1}{2}m(\sqrt{3}-1)y$$
$$+\sinh\tfrac{1}{2}m\{(\sqrt{3}-1)z-2\alpha\}\cosh\tfrac{1}{2}m(\sqrt{3}+1)y;$$

and then the free surface conditions that

$$l\frac{\partial\Phi}{\partial z}=\Phi,$$

where $z=h$, for all values of y, is satisfied if

$$\begin{aligned} ml &= \tanh m(h-\alpha)\\ &=(\sqrt{3}+1)\tanh\tfrac{1}{2}m\{(\sqrt{3}-1)h-2\alpha\}\\ &=(\sqrt{3}-1)\tanh\tfrac{1}{2}m\{(\sqrt{3}+1)h+2\alpha\};\end{aligned}$$

or, putting $m(h-\alpha)=\gamma$,

$$\begin{aligned} ml=\tan h\gamma &=(\sqrt{3}+1)\tanh\left\{\gamma-\tfrac{1}{2}(3-\sqrt{3})mh\right\}\\ &=(\sqrt{3}-1)\tanh\left\{\tfrac{1}{2}(3+\sqrt{3})mh-\gamma\right\}.\end{aligned}$$

From these conditions we find that

$$2 \coth \gamma = \coth \frac{1}{2}(3-\surd 3)\,mh + \coth \frac{1}{2}(3+\surd 3)\,mh,$$

or $\tanh \gamma$ is an harmonic mean between

$$\tanh \frac{1}{2}(3-\surd 3)\,mh \text{ and } \tanh \frac{1}{2}(3+\surd 3)\,mh,$$

and by elimination of γ we obtain the period equation

$$\left\{\coth \frac{1}{2}(3-\surd 3)\,mh + \coth \frac{1}{2}(3+\surd 3)\,mh\right\}^2$$
$$+\surd 3\left\{\coth^2 \frac{1}{2}(3-\surd 3)\,mh - \coth^2 \frac{1}{2}(3+\surd 3)\,mh\right\} = 4,$$

equivalent to

$$(2-\surd 3)\cosh(3+\surd 3)\,mh + (2+\surd 3)\cosh(3-\surd 3)\,mh - \cosh(2mh\surd 3) - 3 = 0.$$

But on investigation it will be found that the only real root of this equation in mh is $mh = 0$, repeated four times, so that progressive waves of this type in a channel of 120° are unstable.

To represent such an unstable state of wave motion, let us change all the hyperbolic functions in Φ into the corresponding circular functions, and put

$$\phi = \Phi \cosh \surd 2\,(mx - nt),$$

in order that the equation of continuity may be satisfied.

Then we have

$$\Phi = \sin m(z-\alpha)\cos my - \sin \frac{1}{2} m(y\surd 3 + z + 2\alpha)\cos \frac{1}{2} m(y - z\surd 3)$$
$$+ \sin \frac{1}{2} m(y\surd 3 - z - 2\alpha)\cos \frac{1}{2} m(y + z\surd 3);$$

so that

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = -2\Phi.$$

Then we shall find

(I) where $y = z\surd 3$, $\dfrac{\partial\Phi}{\partial y} = \surd 3\,\dfrac{\partial\Phi}{\partial z}$;

(II) where $y = -z\surd 3$, $\dfrac{\partial\Phi}{\partial y} = -\surd 3\,\dfrac{\partial\Phi}{\partial z}$;

so that the boundary conditions are satisfied.

Also, the conditions at the free surface $z = h$ are satisfied if

$$ml = \tan m(h-\alpha)$$
$$= (\surd 3 + 1)\tan \frac{1}{2} m\{(\surd 3 - 1)h - 2\alpha\}$$
$$= (\surd 3 - 1)\tan \frac{1}{2} m\{(\surd 3 + 1)h + \alpha\};$$

or,
$$\begin{aligned} ml &= \tan\gamma \\ &= (\surd 3 + 1)\tan\left\{\gamma - \tfrac{1}{2}(3 - \surd 3)\,mh\right\} \\ &= (\surd 3 - 1)\tan\left\{\tfrac{1}{2}(3 + \surd 3)\,mh - \gamma\right\}, \end{aligned}$$
if we write Φ in the form
$$\begin{aligned} \Phi &= \sin m(z - \alpha)\cos my \\ &+ \sin\tfrac{1}{2}m\{(\surd 3 - 1)z - 2\alpha\}\cos\tfrac{1}{2}m(\surd 3 + 1)y \\ &- \sin\tfrac{1}{2}m\{(\surd 3 + 1)z + 2\alpha\}\cos\tfrac{1}{2}m(\surd 3 - 1)y. \end{aligned}$$

As before, we shall find that $\tan\gamma$ is an harmonic mean between
$$\tan\tfrac{1}{2}(3 - \surd 3)\,mh \text{ and } \tan\tfrac{1}{2}(3 + \surd 3)\,mh,$$
and that the period equation is
$$\begin{aligned} &\surd 3\left\{\cot^2\tfrac{1}{2}(3 - \surd 3)\,mh - \cot^2\tfrac{1}{2}(3 + \surd 3)\,mh\right\} \\ &- \left\{\cot\tfrac{1}{2}(3 - \surd 3)\,mh + \cot\tfrac{1}{2}(3 + \surd 3)\,mh\right\}^2 = 4, \end{aligned}$$
or
$$(2 - \surd 3)\cos(3 + \surd 3)\,mh + (2 + \surd 3)\cos(3 - \surd 3)\,mh - \cos(2mh\surd 3) - 3 = 0,$$
an equation with an infinite number of real roots.

This last value of Φ will be useful in attempting to investigate the *bore* or tidal *wave* in a river.

21. *General Wave Motion across a Channel with Plane Sides Sloping at Any Angle.*

Putting, as before,
$$u = z + iy = r(\cos\vartheta + i\sin\vartheta), \text{ and } w = \phi + i\psi,$$
and supposing the sides to slope equally at an angle $\alpha = \pi/2n$ to the horizon, let us attempt the general solution by putting
$$\begin{aligned} w = P_0\cos(u + \alpha_0) &+ P_1\cos(\beta_u + \alpha_1) \\ &+ P_2\cos(\beta_u^2 + \alpha_2) + \ldots + P_{2n-1}\cos(\beta_u^{2n-1} + \alpha_{2n-1}), \end{aligned}$$
where $\beta^{2n} = -1$, so that $\beta = e^{i\alpha}$, $\beta^n = i$.

Then we must determine the P's and α's, so as to satisfy the boundary conditions that $\psi = 0$ or w is real at the sides $\vartheta = \pm(n - 1)\alpha$. The form will be different according as n is odd or even.

A. When n is odd, we must have the P's all real, and

$$P_0 = P_2 = P_4 = \ldots\ldots = A, \text{ suppose};$$
$$P_1 = P_3 = P_5 = \ldots\ldots = B, \text{ suppose};$$

also,

$$\alpha_0 = -\alpha_2 = \alpha_4 = -\alpha_6 = \ldots\ldots = \gamma, \text{ a real quantity};$$
$$\alpha_1 = -\alpha_3 = \alpha_5 = -\alpha_7 = \ldots\ldots = i\delta, \text{ an imaginary.}$$

Hence

$$w = A\{\cos(u+\gamma) + \cos(\beta^2 u - \gamma) + \cos(\beta^4 u + \gamma) + \ldots\ldots\}$$
$$+ B\{\cos(\beta u + i\delta) + \cos(\beta^3 u - i\delta) + \cos(\beta^5 u + i\delta) + \ldots\ldots\};$$

or, as it may be written,

$$w = A\{\cos(u+\gamma) + \cos(\beta^2 u - \gamma) + \cos(\beta^4 u + \gamma) + \ldots\ldots\}$$
$$+ B\{\cosh(u+\delta) + \cosh(\beta^2 u - \delta) + \cosh(\beta^4 u + \delta) + \ldots\ldots\},$$

involving, however, only three disposable constants, A/B, γ and δ.

When $n = 3$, this agrees with the expressions previously obtained, and the three equations of condition at the free surface gave the period equation for mh, the equation for l, and the equation for γ or δ, with A or B alternately zero.

But if we attempt to satisfy the conditions at the free surface with $n = 5, 7, \ldots\ldots$, we shall have more equations to satisfy than the disposable constants mh, ml, A/B, γ and δ, so that the free surface conditions cannot be satisfied.

Separating w into its real and imaginary parts, we find

$$\phi = A\cos(z+\gamma)\cosh y + B\cosh(z+\delta)\cos y$$
$$+ A\cot(z\cos 2\alpha - y\sin 2\alpha - \gamma)\cosh(z\sin 2\alpha + y\cos 2\alpha)$$
$$+ B\cosh(z\cos 2\alpha - y\sin 2\alpha - \delta)\cos(z\sin 2\alpha + y\cos 2\alpha);$$

. . . .

$$\psi = -A\sin(z+\gamma)\sinh y + B\sinh(z+\delta)\sin y$$
$$- A\sin(z\cos 2\alpha - y\sin 2\alpha - \gamma)\sinh(z\sin 2\alpha + y\cos 2\alpha)$$
$$+ B\sinh(z\cos 2\alpha - y\sin 2\alpha - \delta)\sin(z\sin 2\alpha + y\cos 2\alpha).$$

B. When n is even, the boundary conditions lead to

$$P_0 = P_4 = P_8 = \ldots\ldots = A + iB,$$
$$P_2 = P_6 = P_{10} = \ldots\ldots = A - iB,$$
$$P_1 = P_5 = P_9 = \ldots\ldots = C,$$
$$P_3 = P_7 = P_{11} = \ldots\ldots = D,$$

and all the α's must vanish. Then

$$\begin{aligned} w = & (A + iB)(\cos u + \cos \beta^4 u + \cos \beta^8 u + \ldots) \\ & + (A - iB)(\cos \beta^2 u + \cos \beta^6 u + \cos \beta^{10} u + \ldots) \\ & + C(\cos \beta u + \cos \beta^5 u + \cos \beta^9 u + \ldots) \\ & + D(\cos \beta^3 u + \cos \beta^7 u + \cos \beta^{11} u + \ldots), \end{aligned}$$

involving three arbitrary constants, the ratios of A, B, C, D, so that we have not enough disposable constants to satisfy the free surface conditions for an even number greater than 2.

We may suppose the sides of the channel to slope to the horizon at any angles which are the same or different multiples of an n^{th} part of a right angle, and determine the P's and α's from the above general expression for w; thus, for a channel the sides of which slope at 60° to the horizon we shall find

$$\begin{aligned} w = & A\{\cos(u + i\gamma) + \cos(\beta^2 u - i\gamma) + \cos(\beta^4 u + i\gamma)\} \\ & + B\{\cosh(u + i\delta) + \cosh(\beta^2 u - i\delta) + \cosh(\beta^4 u + i\delta)\}, \end{aligned}$$

where $\beta^6 = -1$; but in this, as in the other cases, the free surface conditions cannot be satisfied.

Again, if one side slopes at 30° and the other at 60°, we shall find

$$\begin{aligned} w = & (A + iB)\cos u + E\cos \beta u + (A - iB)\cos \beta^2 u \\ & + (C + iD)\cos \beta^3 u + F\cos \beta^4 u + (C - iD)\cos \beta^5 u. \end{aligned}$$

For, if $\vartheta = 2\alpha$, where $\alpha = \frac{1}{6}\pi$, then $u = r\beta^2$, and

$$\begin{aligned} w = & (A + iB)\cos r\beta^2 + E\cos r\beta^3 + (A - iB)\cos r\beta^4 \\ & + (C + iD)\cos r\beta^5 + F\cos r\beta^6 + (C - iD)\cos r\beta^7, \end{aligned}$$

which is real, since β^2 and $-\beta^4$, β^5 and β^7 are conjugate imaginaries.

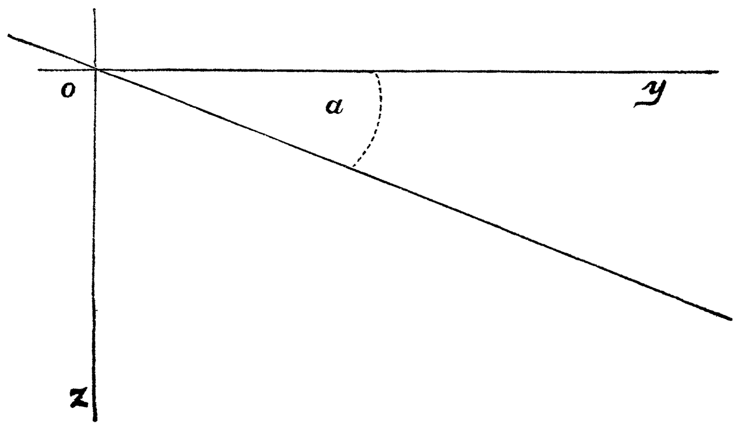
Again, if $\vartheta = -\alpha$, $u = r\beta^{-1}$, and

$$\begin{aligned} w = & (A + iB)\cos r\beta^{-1} + E\cos r + (A - iB)\cos r\beta \\ & + (C + iD)\cos r\beta^2 + E\cos r\beta^3 + (C - iD)\cos r\beta^4, \end{aligned}$$

which also is real, and therefore $\psi = 0$.

22. *Waves against a Uniformly Sloping Shore.*

Let us attempt in a similar manner to determine Kirchoff's general expressions for wave motion against a beach sloping uniformly at an angle α to the horizon (Gesammelte Abhandlungen, p. 431).



Suppose, now, that the axis of z is drawn vertically downwards, and let

$$u = y + iz = r(\cos\vartheta + i\sin\vartheta),$$

so that $\vartheta = \alpha$ at the surface of the shore.

If we put

$$w = \phi + i\psi = \sum_{p=0}^{p=2n-1} P_p \exp i(\beta^{\wedge}u + \alpha_p),$$

where $\alpha = \pi/2n$, then we must have $\psi = 0$, and therefore w real, when $\vartheta = \alpha$, and consequently $u = r\beta$, where $\beta^{2n} = -1$, $\beta^n = i$. Then

$$\begin{aligned} w &= \Sigma P_p \exp i(r\beta^{p+1} + \alpha_p) \\ &= \Sigma P_p \exp(r\beta^{n+p+1} + i\alpha_p), \end{aligned}$$

and for this to be real, we must have $P_{2n-1} = 0$, and

P_0 and P_{2n-2} conjugate imaginaries, as also $i\alpha_0$ and $i\alpha_{2n-2}$,
P_1 " P_{2n-3} " " " $i\alpha_1$ " $i\alpha_{2n-3}$,
P_2 " P_{2n-4} " " " $i\alpha_2$ " $i\alpha_{2n-4}$,
. . . .

P_{n-1} is real, and also $i\alpha_{n-1}$.

We may then write $P_0 \exp i\alpha_0$ and $P_{2n-2} \exp i\alpha_{2n-2}$ in the form $A_0 \exp i\gamma_0$ and $A_0 \exp(-i\gamma_0)$ without loss of generality, and similar expressions for $P_1 \exp i\alpha_1, \ldots.,$ and replace $P_{n-1} \exp i\alpha_{n-1}$ by A_{n-1}, so that now

$$\begin{aligned} w = \sum_{p=0}^{p=n-2} A_p \,[&\exp\{-r\sin(\theta + p\alpha) + ir\cos(\vartheta + p\alpha) + i\gamma_p\} \\ &+ \exp\{-r\sin(\vartheta + \overline{2n-p-2}.\alpha) + ir\cos(\vartheta + \overline{2n-p-2}.\alpha) - i\gamma_p\}] \\ &+ A_{n-1}\exp\{-r\sin(\vartheta + \overline{n-1}.\alpha) + ir\cos(\vartheta + \overline{n-1}.\alpha)\}, \end{aligned}$$

giving ϕ and ψ, satisfying the equation of continuity, and satisfying the boundary condition that $\psi = 0$ and w therefore real when $\vartheta = \alpha = \pi/2n$.

At the free surface $\vartheta = 0$ we must have

$$l\frac{\partial\phi}{\partial z} = -\phi, \text{ or } l\frac{\partial\phi}{r\partial\vartheta} = -\phi,$$

for all values of r.

But
$$\frac{\partial\phi}{r\partial\vartheta} = -\frac{\partial\psi}{\partial z},$$

so that we must have
$$l\frac{\partial\psi}{\partial r} = \phi$$

for all values of r at the free surface $\vartheta = 0$; and we may put $\vartheta = 0$ in ϕ and ψ before differentiation, which simplifies the calculations considerably.

Now, putting $\vartheta = 0$,

$$\begin{aligned}
\phi = \sum_{p=0}^{p=n-2} A_p \{&\exp(-r\sin p\alpha)\cos(r\cos p\alpha + \gamma_p) \\
&+ \exp(-r\sin \overline{p+2}.\alpha)\cos(r\cos \overline{p+2}.\alpha + \gamma_p)\} \\
&+ A_{n-1}\exp(-r\sin \overline{n-1}.\alpha)\cos(r\cos \overline{n-1}.\alpha), \\
\psi = \Sigma A_p \{&\exp(-r\sin p\alpha)\sin(r\cos p\alpha + \gamma_p) \\
&- \exp(-r\sin \overline{p+2}.\alpha)\sin(r\cos \overline{p+2}.\alpha + \gamma_p)\} \\
&+ A_{n-1}\exp(-r\sin \overline{n-1}.\alpha)\sin(r\cos \overline{n-1}.\alpha), \\
\frac{\partial\psi}{\partial r} = \Sigma A_p \{&\exp(-r\sin p\alpha)\cos(r\cos p\alpha + p\alpha + \gamma_p) \\
&- \exp(-r\sin \overline{p+2}.\alpha)\cos(r\cos \overline{p+2}.\alpha + \overline{p+2}.\alpha + \gamma_p)\} \\
&+ A_{n-1}\exp(-r\sin \overline{n-1}.\alpha)\cos(r\cos \overline{n-1}.\alpha + \overline{n-1}.\alpha).
\end{aligned}$$

Equating coefficients in ϕ and $\frac{\partial\psi}{\partial r}$ of $\exp(-r\sin p\alpha)$, we have

$$\begin{aligned}
&A_0\cos(r + \gamma_0) \\
&= A_0\cos(r + \gamma_0), \\
&A_1\cos(r\cos\alpha + \gamma_1) \\
&= A_1\cos(r\cos\alpha + \alpha + \gamma_1), \\
&A_2\cos(r\cos 2\alpha + \gamma_2) + A_0\cos(r\cos 2\alpha + \gamma_0) \\
&= A_2\cos(r\cos 2\alpha + 2\alpha + \gamma_2) - A_0\cos(r\cos 2\alpha + 2\alpha + \gamma_0), \\
&A_3\cos(r\cos 3\alpha + \gamma_3) + A_1\cos(r\cos 3\alpha + \gamma_1) \\
&= A_3\cos(r\cos 3\alpha + 3\alpha + \gamma_3) - A_1\cos(r\cos 3\alpha + 3\alpha + \gamma_1), \\
&A_4\cos(r\cos 4\alpha + \gamma_4) + A_2\cos(r\cos 4\alpha + \gamma_2) \\
&= A_4\cos(r\cos 4\alpha + 4\alpha + \gamma_4) - A_2\cos(r\cos 4\alpha + 4\alpha + \gamma_2), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$A_{n-1}\cos(r\cos n-1\,.\,\alpha)+A_{n-3}\cos(r\cos n-1\,.\,\alpha+\gamma_{n-3})$$
$$=A_{n-1}\cos(r\cos n-1\,.\,\alpha+n-1\,.\,\alpha)-A_{n-3}\cos(r\cos n-1\,.\,\alpha+n-1\,.\,\alpha+\gamma_{n-3}),$$
$$A_{n-2}\cos\gamma_{n-2}$$
$$=-A_{n-2}\cos(n\alpha+\gamma_{n-2})$$
$$=\quad A_{n-2}\sin\gamma_{n-2}.$$

The first equation is satisfied identically, a is satisfied by $A_1=0$, which makes all the odd A's vanish, and then the even A's are determined by

$$A_2\sin(r\cos 2\alpha+\alpha+\gamma_2)\sin\alpha=A_0\sin\left(r\cos 2\alpha+\alpha+\gamma_0-\frac{1}{2}\pi\right)\cos\alpha,$$

$$A_4\sin(r\cos 4\alpha+2\alpha+\gamma_4)\sin 2\alpha=A_2\sin\left(r\cos 4\alpha+2\alpha+\gamma_2-\frac{1}{2}\pi\right)\cos 2\alpha,$$

. . . .

giving
$$A_2\sin\alpha=A_0\cos\alpha,$$
$$A_4\sin 2\alpha=A_2\cos 2\alpha,$$
. . . .

and
$$\gamma_2=\gamma_0-\frac{1}{2}\pi,$$
$$\gamma_4=\gamma_2-\frac{1}{2}\pi=\gamma_0-\pi.$$
. . . .

The form of the result is different according as n is odd or even.

A. If n is odd $=2m+1$, then the final equations are

$$A_{2m}\cos(r\cos 2m\alpha)+A_{2n-2}\cos(r\cos 2m\alpha+\gamma_{2m-2})$$
$$=A_{2m}\cos(r\cos 2m\alpha+2m\alpha)-A_{2m-2}\cos(r\cos 2m\alpha+2m\alpha+\gamma_{2m-2});$$

or,

$$A_{2m}\sin(r\cos 2m\alpha+m\alpha)\sin m\alpha=A_{2m-2}\sin\left(r\cos 2m\alpha+m\alpha+\gamma_{2m-2}-\frac{1}{2}\pi\right)\cos m\alpha,$$

giving
$$A_{2m}\sin m\alpha=A_{2m-2}\cos m\alpha,$$

and
$$\gamma_{2m-2}=\frac{1}{2}\pi;$$

also,
$$A_{n-2}=A_{2m-1}=0.$$

Then
$$\gamma_{2m-4}=\pi,$$
$$\gamma_{2m-6}=\frac{3}{2}\pi,$$
$$\gamma_{2m-8}=2\pi,$$
. . . .

B. If n is even, $=2m$, then A_{n-2} does not vanish; so that $\tan\gamma_{n-2}=1$, $\gamma_{n-2}=\frac{1}{4}\pi$, whence the other values of γ are determined.

Suppose, for instance, $n = 4$; then $\alpha = \frac{1}{8}\pi$, and

$$\gamma_2 = \frac{1}{4}\pi,\ \gamma_0 = \frac{3}{2}\pi;$$

also,
$$A_2 = A_0 \cot\alpha = A_0(\sqrt{2}+1),$$
agreeing with Kirchoff's results.

Again, suppose $n = 5$, $\alpha = \frac{1}{10}\pi$;

then
$$\gamma_2 = \frac{1}{2}\pi,\ \gamma_0 = \pi;$$

also,
$$A_2 = A_0 \cot\alpha,$$
$$A_4 = A_2 \cot 2\alpha = A_0 \cot\alpha \cot 2\alpha.$$

For
$$n = 2,\ \gamma_0 = \frac{1}{4}\pi,$$

and for
$$n = 3,\ \gamma_0 = \frac{1}{2}\pi,\ A_2 = A_0 \cot\alpha = A_0\sqrt{3},$$
agreeing with the results given by Kirchoff (Gesammelte Abhandlungen, p. 434).

23. *Algebraical Solution of Waves against a Sloping Beach.*

of continuity and the boundary condition that
tting

$$\begin{aligned}\psi + i\phi = r^n &(\cos n\vartheta + i \sin n\vartheta)\\ &+ A_1 a r^{n-1}\{\cos(n-1.\vartheta+\alpha) + i\sin(n-1.\vartheta+\alpha)\},\\ &+ A_2 a^2 r^{n-2}\{\cos(n-2.\vartheta+2\alpha) + i\sin(n-2.\vartheta+2\alpha)\},\\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+ A_n a^n(\cos n\alpha + i\sin n\alpha),\end{aligned}$$

equivalent to

$$\psi + i\phi = u^n + A_1 a\beta u^{n-1} + A_2 a^2\beta^2 u^{n-2} + \ldots\ldots,$$

an algebraical function of

$$u = y + iz = r(\cos\vartheta + i\sin\vartheta);$$

and then $\psi = 0$ when $\vartheta = \alpha$, since $n\alpha = \frac{1}{2}\pi$.

Put
$$\begin{aligned}\psi = r^n \cos n\vartheta &- A_1 a r^{n-1}\cos(n-1.\vartheta+\alpha)\\ &+ A_2 a^2 r^{n-2}\cos(n-2.\vartheta+2\alpha) - A_3 a^3 r^{n-3}\cos(n-3.\vartheta+3\alpha)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

and then at the free surface $\vartheta = 0$ we shall have

$$a\frac{\partial\psi}{\partial r} = \phi$$

for all values of r, if

$$\begin{aligned} A_1 \sin\alpha \;\; &= n, \\ A_2 \sin 2\alpha &= (n-1) A_1 \cos\alpha, \\ A_3 \sin 3\alpha &= (n-2) A_2 \cos 2\alpha, \\ &\cdots\cdots\cdots\cdots \\ A_n \sin n\alpha &= A_{n-1} \cos (n-1)\alpha; \end{aligned}$$

n equations for determining $A_1, A_2, A_3, \ldots . A_n$.

Here the motion increases indefinitely with y and z, so we must seek to determine possible boundaries to limit the motion, and to contain the liquid in a finite cylinder.

Suppose, for instance, $n = 3$; then for $\alpha = \frac{1}{6}\pi$,

$$A_1 = 6, \quad A_2 = 12, \quad A_3 = 6;$$

so that

$$\psi + i\phi = \left(r^{i\vartheta} - \frac{ae^{i\alpha}}{\sin\alpha}\right)^3 + \text{const.},$$

and with a new origin

$$\psi + i\phi = u^3 + \text{const.},$$

the algebraical motion previously investigated in a channel of 120°.

Again, suppose $n = 4$, $\alpha = \frac{1}{8}\pi$; then

$$A_1 \sin\alpha = 4, \quad A_2 \sin 2\alpha = 3A_1 \cos\alpha, \quad A_3 \sin 3\alpha = 2A_2 \cos 2\alpha, \quad A_4 = A_3 \cos 3\alpha,$$

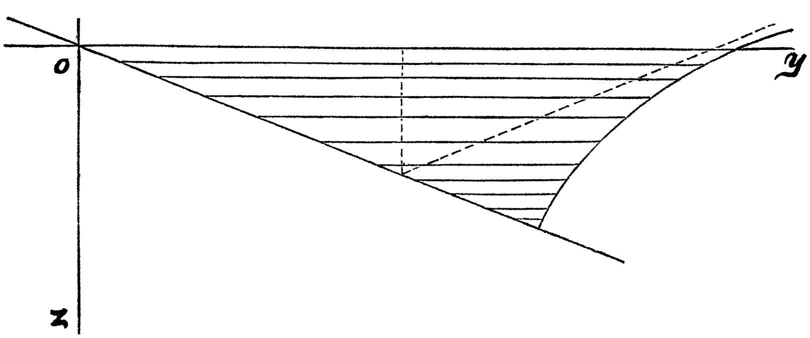
giving

$$A_1 = 4 \operatorname{cosec}\alpha, \quad A_2 = 6 \operatorname{cosec}^2\alpha, \quad A_3 = 24 \operatorname{cosec}\alpha, \quad A_4 = 24;$$

so that

$$\psi = r^4 \cos 4\vartheta + 4ar^3 \frac{\cos(3\vartheta + \alpha)}{\sin\alpha} = 6a^2r^2 \frac{\cos(2\vartheta + 2\alpha)}{\sin^2\alpha} + 24a^3r \frac{\cos(\vartheta + 3\alpha)}{\sin\alpha},$$

one factor of which must be $r\cos(\vartheta + 3\alpha)$, and the other factor equated to zero will give the equation of a curve which can be used to limit the motion, which therefore takes place across a cylinder, the section of which is as in the figure.



24. *Wave Motion in a Cone.*

If we put
$$\phi = zr^n \sin n\vartheta,$$
employing the cylindrical co-ordinates r, ϑ, z, then the equation of continuity
$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{r^2\partial\vartheta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$
is satisfied; also, at the free surface $z = h$,
$$h\frac{\partial\phi}{\partial z} = \phi;$$
so that the length of the equivalent simple pendulum of the wave motion is h.

We must now seek to determine the shape of a vessel which will contain liquid having the above given motion.

Supposing the vessel is of revolution, then along a meridian section, the axis being vertical,
$$\frac{dz}{\dfrac{\partial\phi}{\partial z}} = \frac{dr}{\dfrac{\partial\phi}{\partial r}}$$
or,
$$\frac{dz}{r^n \sin n\vartheta} = \frac{dr}{nr^{n-1}\sin n\vartheta};$$
or,
$$nzdz = rdr;$$
so that
$$nz^2 = r^2 + \text{const.}$$
is the equation of a meridian section, which is therefore in general a hyperbola, except when the constant is zero, when it degenerates into two straight lines.

The vessel is therefore a hyperboloid of revolution, including a cone of vertical angle $2\tan^{-1}\sqrt{n}$ as a particular case.

If
$$n = 1,\ \phi = yz,$$
the same as for the algebraical motion across a channel of 90°.

The stream lines are rectangular hyperbolas, so that the boundary may be supposed a horizontal cylinder on a vertical cone of which the vertical sections are rectangular hyperbolas.

If
$$n = 2,\ \phi = xyz,$$
and the vertical angle of the cone is $2\tan^{-1}\sqrt{z}$.

The differential equations of the stream lines are now
$$xdx = ydy = zdz,$$
so that we may suppose the containing vessel, in its most general form, a hyperboloid whose equation is
$$ax^2 + by^2 + cz^2 = \text{constant},$$
subject to the condition
$$a + b + c = 0.$$

Generally, for any value of n, we may suppose vertical meridian plane diaphragms, given in position by

$$\cos n\vartheta = 0,$$

to be introduced without disturbing the motion.

Also, since $\dfrac{\partial^3\phi}{\partial z^3} = 0$, this kind of wave motion will be unaffected by any capillarity of the free surface.

25. *Wave Motion in a Cylinder.*

In seeking the particular solutions of the above equation of continuity in cylindrical co-ordinates, if we assume that ϕ has a factor $\sin n\vartheta$, so that the wave motion may be limited by vertical meridian diaphragms, given by $\cos n\vartheta = 0$, then

$$r^2 \frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + r^2\frac{\partial^2\phi}{\partial z^2} - n^2\phi = 0;$$

and if we further assume that ϕ has a factor $\cosh(kz + \beta)$, the other factor being a function of r only, then

$$r^2 \frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + (k^2r^2 - n^2)\phi = 0,$$

Bessel's differential equation; so that we may put

$$\phi = \cosh(kz + \beta)\, J_n(kr)\sin n\vartheta,$$

as a type of the particular solutions of the equation of continuity.

A single term of this nature is appropriate for determining wave motion inside a vertical circular cylinder.

Since $\dfrac{\partial\phi}{\partial z} = 0$ at the base, $z = 0$, therefore $\beta = 0$; also, at the free surface, $z = h$,

$$l\frac{\partial\phi}{\partial z} = \phi,$$

or

$$kl = \coth kh;$$

and k is determined from the condition that at the cylindrical boundary $r = a$, $\dfrac{\partial\phi}{\partial r} = 0$, or

$$J_n'(ka) = 0.$$

This kind of wave motion in a cylinder has been completely investigated by Lord Rayleigh.

When $n = i + \dfrac{1}{2}$, where i denotes an integer, then the corresponding Bessel's function is an algebraical and trigonometrical function of r, so that we can obtain a corresponding solution for wave motion in the space bounded by the cylinder and two diametral planes inclined at an angle $2\pi/(2i+1)$; *e. g.* $\dfrac{2}{3}\pi, \dfrac{2}{5}\pi, \ldots.$

INDEX.